The Realizability of Multiport Structures Obtained by Imbedding a Tunnel Diode in a Lossless Reciprocal Network

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Necessary and sufficient conditions are presented for the realization of the short-circuit admittance matrix or open-circuit impedance matrix of the most general n-port structures characterized by such matrices obtained by imbedding a tunnel diode, represented by a parallel combination of a capacitor and a negative resistor, in a finite lossless reciprocal network. Techniques for realizing prescribed immittance matrices are included.

I. INTRODUCTION

It is generally well known that the tunnel diode possesses a small-signal equivalent circuit that can often be approximated by a parallel combination of a capacitor and a negative resistor. This model has been used extensively in the study of gain-bandwidth relations and optimum synthesis procedures for specific amplifier configurations.[1-5] It has also been used to derive bounds on the natural frequencies obtained by imbedding the tunnel diode in a passive network.[6,7]

The purpose of this paper is to present necessary and sufficient conditions for the realization of the short-circuit admittance matrix or open-circuit impedance matrix of the most general n-port structures characterized by such matrices obtained by imbedding a tunnel diode, represented by the above mentioned model, in a lossless reciprocal network.

The properties of the short-circuit admittance matrix are considered also by another writer.[8] With the exception of certain remarks of a tutorial nature, the arguments, results, synthesis techniques, and basic approach to the problem presented here are quite different from that in Ref. 8. In particular, it is not assumed here that the short-circuit admittance matrix of the $(n + 1)$-port lossless network invariably exists.

Also, the necessary and sufficient conditions are stated directly in terms of the $n \times n$ short-circuit admittance matrix and its even part. They do not involve a knowledge of the short-circuit admittance matrix obtained when the tunnel diode is short-circuited.

II. DESCRIPTION OF THE STRUCTURE TO BE CONSIDERED

The basic structure under consideration is shown in Fig. 1, in which the $(n + 1)$-port network is assumed to be a lossless reciprocal configuration containing inductors, capacitors, and ideal transformers. Port $(n + 1)$ is terminated with a unit capacitor and unit resistor in parallel. This involves no loss of generality since a similar termination with other values of positive capacitance and/or resistance (positive or negative) can be treated with the aid of simple transformations which are explicitly stated in Section VII. The overall network is restricted initially in that the symmetric positive-real short-circuit admittance matrix $\mathbf{Y}(s)$, relating the port currents and voltages at ports $(1, 2, \cdots, n)$, is assumed to exist. The realizability conditions for the open-circuit impedance matrix $\mathbf{Z}(s)$ can be obtained in a manner similar to that to be described for $\mathbf{Y}(s)$ and are stated in Section VII.

The $(n + 1)$-port lossless network is characterized by the regular para-unitary scattering matrix $\hat{\mathbf{S}}(s)$ or by the short-circuit admittance matrix $\hat{\mathbf{Y}}(s)$, when it exists. We initially assume that $\hat{\mathbf{Y}}(s)$ does exist and consider in a subsequent section the case in which $\hat{\mathbf{Y}}(s)$ does not exist.

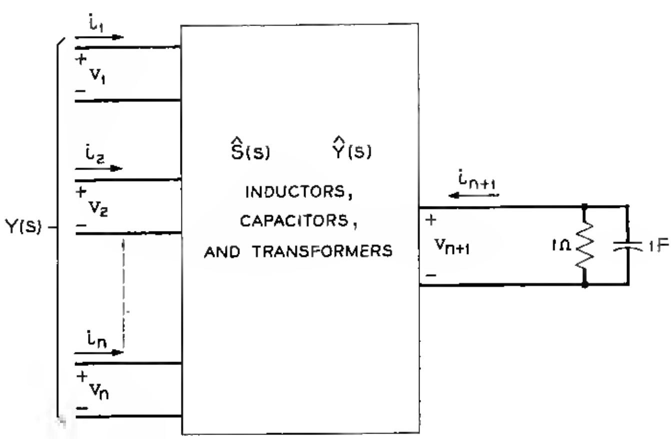

Fig. 1 — Most general structure defining $\mathbf{Y}(s)$.

III. NECESSARY CONDITIONS FOR THE REALIZATION OF $\mathbf{Y}(s)$ WHEN $\hat{\mathbf{Y}}(s)$ EXISTS AND THE EVEN PART OF $\mathbf{Y}(s)$ IS NOT A MATRIX OF CONSTANTS

The necessary and sufficient conditions for the realization of $\hat{\mathbf{Y}}(s)$ are, of course, well known.

It is also well known that†

$$\mathbf{Y} = \mathbf{Y}_{11} - \mathbf{Y}_{12}\mathbf{Y}_{12}{}^t \frac{1}{Y_{22} + s + 1} \tag{1}$$

where the matrices in (1) are defined by the following partition of $\hat{\mathbf{Y}}(s)$:

$$\hat{\mathbf{Y}} = \begin{array}{cc} \begin{array}{cc} n & 1 \end{array} & \\ \begin{bmatrix} \mathbf{Y}_{11} & \mathbf{Y}_{12} \\ \mathbf{Y}_{12}{}^t & Y_{22} \end{bmatrix} & \begin{array}{c} n \\ 1 \end{array} \end{array}. \tag{2}$$

The arguments to be presented center about a study of $\mathbf{Y}_e$, the even part of the matrix $\mathbf{Y}$. This matrix is given by

$$\begin{aligned} \mathbf{Y}_e &= \tfrac{1}{2}[\mathbf{Y}(s) + \mathbf{Y}(-s)] \\ &= -\mathbf{Y}_{12}\mathbf{Y}_{12}{}^t \frac{1}{[Y_{22} + s + 1][-Y_{22} - s + 1]}. \end{aligned} \tag{3}$$

It is convenient to introduce the notation: $Y_{22} = d^{-1}n_{22}$, $\mathbf{Y}_{12} = d^{-1}\mathbf{N}_{12}$ where d is an even polynomial, n_{22} is an odd polynomial and $\mathbf{N}_{12}$ is a matrix of odd polynomials, with the understanding that d, n_{22} and every element in $\mathbf{N}_{12}$ may have a common simple zero at the origin. In this way it is unnecessary to treat separately the cases in which d is even or d is odd. Accordingly,

$$\mathbf{Y}_e = -\mathbf{N}_{12}\mathbf{N}_{12}{}^t \frac{1}{[n_{22} + (s + 1)\, d][n_{22}(-s) + (-s + 1)\, d]}. \tag{4}$$

Note that the polynomial $[n_{22} + (s + 1)\, d]$ can be assumed to be strictly Hurwitz except possibly for a simple zero at the origin, since n_{22} and d can be assumed to be relatively prime except possibly for a simple common zero at the origin.

It is convenient to treat separately the cases in which $\mathbf{Y}_e$ is or is not a matrix of constants.

† The superscript t denotes matrix transposition.

Consider the following

Definition:

The matrix $\mathbf{Y}_e$ *is said to be in standard form if and only if*

$$\mathbf{Y}_e = -\mathbf{U}\mathbf{U}^t \frac{1}{v(s)v(-s)}$$

where $v(s)$ *is a positive coefficient polynomial which is strictly Hurwitz except possibly for a simple zero at the origin and* $\mathbf{U}^t = [u_1, u_2, \cdots, u_n]$ *is a matrix of odd real polynomials with the property that there is no factor* $\eta(s)\eta(-s)$ *common to all the* u_i *such that* $\eta^2(s)\eta^2(-s)$ *divides* $v(s)v(-s)$ *where* $\eta(s)$ *is a strict Hurwitz polynomial. The polynomials* v_e *and* v_o *are respectively the even and odd parts of* $v(s)$.

In Section IV the following result is proved.

Theorem 1:

A rational positive-real symmetric matrix $\mathbf{Y}(s)$ *with nonconstant even part is realizable as shown in Fig. 1 when* $\hat{\mathbf{Y}}(s)$ *exists only if* $\mathbf{Y}_e$ *is expressible in standard form with* $v(s)$ *such that*†

$$i. \quad k = \left[\frac{v_o}{sv_e}\right]_\infty \geqq 1, \qquad and$$

$$ii. \; If\ k = 1, \qquad [\mathbf{Y}_e]_\infty = \mathbf{0},$$

$$iii. \; If\ k > 1, \qquad \left[\frac{1}{s}\mathbf{Y}\right]_\infty - \frac{k}{k-1}[\mathbf{Y}_e]_\infty .$$

is nonnegative definite.

The case in which $\mathbf{Y}_e$ is a matrix of constants is treated in Section VI.

IV. PROOF OF THEOREM 1

We begin by observing from (4) that $\mathbf{Y}_e$ can be expressed in standard form if $\mathbf{Y}$ is realizable. The problem of factoring a given matrix $\mathbf{Y}_e$ into the required form is discussed in detail in Appendix A.

Assume now that $\mathbf{Y}_e$ is given in standard form and consider the problem of identifying $\mathbf{N}_{12}$, n_{22}, and d in (4). A common factor may have been canceled in the expression for $\mathbf{Y}_e$, and hence an unknown factor must be reinserted before $\mathbf{N}_{12}$, n_{22}, and d can be determined. However,

† Throughout we use the notation $\lim_{s\to\infty} [\cdot] = [\cdot]_\infty$.

the common factor must be of the form $a^2(s) = b(s)b(-s)$ where $b(s)$ is a strict Hurwitz polynomial. Therefore, ignoring a possible minus sign, $a(s) = \eta(s)\eta(-s)$ where $\eta(s)$ is a strict Hurwitz polynomial.

Thus, for some unknown strict Hurwitz $\eta(s)$,

$$\mathbf{N}_{12} \frac{1}{[n_{22} + (s+1)\,d]} = \mathbf{U}\eta(s)\eta(-s)\,\frac{1}{v(s)\eta^2(s)} \tag{5}$$

and

$$\mathbf{N}_{12} = \mathbf{U}\eta(s)\eta(-s) \tag{6}$$

$$n_{22} + (s+1)\,d = v(s)\eta^2(s). \tag{7}$$

In the following we shall denote by η_e and η_o the even and odd parts respectively of η. Equations (6) and (7) read

$$\mathbf{N}_{12} = \mathbf{U}[\eta_e^2 - \eta_o^2] \tag{8}$$

$$n_{22} + (s+1)\,d = v_e(\eta_e^2 + \eta_o^2) + v_o(\eta_e^2 + \eta_o^2) + 2v_o\eta_e\eta_o + 2v_e\eta_e\eta_o\,. \tag{9}$$

Equating even and odd parts of (9) gives

$$\begin{aligned} d &= v_e(\eta_e^2 + \eta_o^2) + 2v_o\eta_e\eta_o \\ n_{22} &= 2v_e\eta_e\eta_o + v_o(\eta_e^2 + \eta_o^2) - s\,d \end{aligned} \tag{10}$$

and therefore

$$Y_{22} = \frac{n_{22}}{d} = \frac{v_o(\eta_e^2 + \eta_o^2) + 2v_e\eta_e\eta_o}{v_e(\eta_e^2 + \eta_o^2) + 2v_o\eta_e\eta_o} - s\,. \tag{11}$$

From (11) it is clear that Y_{22} is realizable provided

$$\left[\frac{1}{s}\,Y_{22}\right]_\infty \geqq 0\,. \tag{12}$$

However,

$$Y_{22} = \frac{n_{22}}{d} = \frac{\dfrac{v_o}{v_e} + \dfrac{2\eta_e\eta_o}{\eta_e^2 + \eta_o^2}}{1 + \dfrac{2v_o}{v_e}\left[\dfrac{\eta_e\eta_o}{\eta_e^2 + \eta_o^2}\right]} - s \tag{13}$$

and since $\eta(s)$ is a Hurwitz polynomial†

$$\left[\frac{1}{s}\,Y_{22}\right]_\infty = \left[\frac{1}{s}\,\frac{v_o}{v_e}\right]_\infty \frac{1}{1+\alpha} - 1 \tag{14}$$

† We have assumed that the degree of v_o exceeds the degree of v_e. It is easy to show that it is impossible to satisfy (12) unless this is so.

where

$$0 \leqq \alpha = \left\{\frac{2v_o}{v_e}\left[\frac{\eta_e\eta_o}{\eta_e^2 + \eta_o^2}\right]\right\}_\infty < \infty . \tag{15}$$

Clearly, it is necessary that

$$k = \left[\frac{1}{s}\frac{v_o}{v_e}\right]_\infty \geqq 1 . \tag{16}$$

4.1 *Derivation of the Inequality Involving* $\mathbf{K}_\infty$ *and* $[\mathbf{Y}_e]_\infty$

Consider now the derivation of a key inequality that must be satisfied by the coefficient matrix $\mathbf{K}_\infty = [(1/s)\mathbf{Y}]_\infty$.

Let the constant matrices $\mathbf{A}_{ij}$ be defined by

$$\mathbf{A}_{ij} = \left[\frac{1}{s}\mathbf{Y}_{ij}\right]_\infty . \tag{17}$$

Then, from (1),

$$\mathbf{K}_\infty = \left[\frac{1}{s}\mathbf{Y}\right]_\infty = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{12}{}^t\frac{1}{A_{22}+1} . \tag{18}$$

But from (8) and (10)

$$\mathbf{Y}_{12} = \frac{1}{d}\mathbf{N}_{12} = \frac{1}{v_e}\mathbf{U}\frac{\left[\dfrac{\eta_e^2 - \eta_o^2}{\eta_e^2 + \eta_o^2}\right]}{1 + \dfrac{2v_o}{v_e}\left[\dfrac{\eta_e\eta_o}{\eta_e^2 + \eta_o^2}\right]}$$

and thus

$$\mathbf{A}_{12} = \left[\frac{1}{sv_e}\mathbf{U}\right]_\infty \frac{\pm 1}{1+\alpha} \tag{19}$$

where α is defined in (15) and the plus or minus sign applies according as the degree of η_e exceeds the degree of η_o or not. Recall from (14) and (16) that

$$A_{22} = \frac{k}{1+\alpha} - 1 . \tag{20}$$

Using (18), (19), and (20)

$$\mathbf{K}_\infty = \mathbf{A}_{11} - \left[\frac{1}{sv_e}\mathbf{U}\right]_\infty\left[\frac{1}{sv_e}\mathbf{U}^t\right]_\infty\frac{1}{k(1+\alpha)} \tag{21}$$

where $\mathbf{A}_{11}$ is unknown. However, if the $(n + 1)$-port lossless network is to be realizable, it is necessary that the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}{}^{t} & A_{22} \end{bmatrix} \tag{22}$$

is nonnegative definite. Assume initially that $A_{22} \neq 0$. We require the following result.

Lemma 1:

A necessary and sufficient condition that $\mathbf{A}$ *is nonnegative definite with* $A_{22} \neq 0$ *is that* $\mathbf{A}'$ *is nonnegative definite where*

$$\mathbf{A}' = \begin{bmatrix} \mathbf{A}_{11} - A_{22}{}^{-1}\mathbf{A}_{12}\mathbf{A}_{12}{}^{t} & \mathbf{0} \\ \mathbf{0} & A_{22} \end{bmatrix}. \tag{23}$$

To prove the lemma, note that $\mathbf{A}' = \mathbf{B}\mathbf{A}\mathbf{B}^{t}$ with

$$\mathbf{B} = \begin{bmatrix} \mathbf{1}_{n} & -A_{22}{}^{-1}\mathbf{A}_{12} \\ \mathbf{0} & 1 \end{bmatrix}$$

where $\mathbf{1}_n$ is the identity matrix of order n.

Thus,

$$\mathbf{A}_{11} - A_{22}{}^{-1}\mathbf{A}_{12}\mathbf{A}_{12}{}^{t}$$

or equivalently

$$\mathbf{A}_{11} - \left[\frac{1}{sv_e}\mathbf{U}\right]_{\infty}\left[\frac{1}{sv_e}\mathbf{U}^{t}\right]_{\infty}\frac{1}{(1 + \alpha)(k - 1 - \alpha)} \tag{24}$$

is required to be nonnegative definite. By combining this result with (21) we find that

$$\mathbf{K}_{\infty} - \frac{1}{k(k - 1 - \alpha)}\left[\frac{1}{sv_e}\mathbf{U}\right]_{\infty}\left[\frac{1}{sv_e}\mathbf{U}^{t}\right]_{\infty} \tag{25}$$

is nonnegative definite. Recalling that $(k - 1 - \alpha)$ is initially assumed to exceed zero, it is clear that (25) is nonnegative definite with $\alpha = 0$. Furthermore, with $\alpha = 0$, (25) can be expressed as

$$\mathbf{K}_{\infty} - \frac{k}{k - 1}[\mathbf{Y}_e]_{\infty} \tag{26}$$

by using the identities

$$\left[\frac{1}{sv_e}\mathbf{U}\right]_\infty = \left[\frac{v_o}{sv_e}\right]_\infty \left[\frac{1}{v_o}\mathbf{U}\right]_\infty \tag{27}$$

$$[\mathbf{Y}_e]_\infty = \left[\frac{1}{v_o}\mathbf{U}\right]_\infty \left[\frac{1}{v_o}\mathbf{U}^t\right]_\infty . \tag{28}$$

When $A_{22} = 0$, as is the case whenever $k = 1$, it is clear from (22) that every element in $\mathbf{A}_{12}$ must vanish if $\mathbf{A}$ is to be nonnegative definite. But from (19)

$$\mathbf{A}_{12} = k\left[\frac{1}{v_o}\mathbf{U}\right]_\infty \frac{\pm 1}{1+\alpha}.$$

Thus, from (28) it is evident that $[\mathbf{Y}_e]_\infty = \mathbf{0}$ when $k = 1$.

This proves Theorem 1. In the next section we prove that if a positive-real matrix satisfies the conditions of Theorem 1, it is realizable as shown in Fig. 1.

V. PROOF OF SUFFICIENCY OF THEOREM I FOR $\mathbf{Y}(s)$ WITH NON-CONSTANT $\mathbf{Y}_e$

Assume that $\mathbf{Y}(s)$ and

$$\mathbf{Y}_e = -\frac{1}{v(s)v(-s)}\mathbf{U}\mathbf{U}^t, \tag{29}$$

in standard form, are prescribed and satisfy Theorem 1.

Let

$$\begin{aligned} Y_{22} &= \frac{n_{22}}{d} = \frac{v_o}{v_e} - s \\ \mathbf{Y}_{12} &= \frac{1}{d}\mathbf{N}_{12} = \frac{1}{v_e}\mathbf{U}. \end{aligned} \tag{30}$$

Then from (30) and (1)

$$\mathbf{Y}_{11} = \mathbf{Y} + \frac{1}{v_e(v_o+v_e)}\mathbf{U}\mathbf{U}^t. \tag{31}$$

Hence $\mathbf{Y}_{11}$, $\mathbf{Y}_{12}$, and Y_{22} satisfy (1). We wish to prove that these sub-matrices defined above lead to a realizable $\hat{\mathbf{Y}}(s)$ given by

$$\hat{\mathbf{Y}}(s) = \begin{bmatrix} \mathbf{Y} + \dfrac{1}{v_e(v_o+v_e)}\mathbf{U}\mathbf{U}^t & \dfrac{1}{v_e}\mathbf{U} \\ \dfrac{1}{v_e}\mathbf{U}^t & \dfrac{v_o}{v_e} - s \end{bmatrix}. \tag{32}$$

First of all, Y_{22} is a realizable driving-point admittance function since $k \geqq 1$.

The submatrix $\mathbf{Y}_{11}$ can be expressed as follows by using (29):

$$\begin{aligned}\mathbf{Y}_{11} &= \mathbf{Y}_o - \frac{1}{v(s)v(-s)}\,\mathbf{U}\mathbf{U}^t + \frac{1}{v_e[v_o + v_e]}\,\mathbf{U}\mathbf{U}^t \\ &= \mathbf{Y}_o - \frac{v_o}{v_e}\,\frac{1}{v(s)v(-s)}\,\mathbf{U}\mathbf{U}^t.\end{aligned} \tag{33}$$

where $\mathbf{Y}_o$ is the odd part of $\mathbf{Y}$. From (33) it is apparent that $\mathbf{Y}_{11}$ is a matrix of odd functions, as it should be. Further, since from (31) $\mathbf{Y}_{11}$ is regular in the right-half plane, it follows that $\mathbf{Y}_{11}$ can have poles only on the $j\omega$ axis. In fact, the finite poles of $\mathbf{Y}_{11}$ are the boundary poles of $\mathbf{Y}$ and the zeros of v_e.

Consider now the residue matrix $\hat{\mathbf{K}}_i$ at a pole of $\hat{\mathbf{Y}}(s)$ which arises from a zero of v_e, say at $s = j\omega_i$, and let the residue matrix of $\mathbf{Y}$ at that pole be $\mathbf{K}_i$. Then†

$$\hat{\mathbf{K}}_i = \begin{bmatrix} \mathbf{K}_i + \dfrac{1}{\dot{v}_e v_o}\,\mathbf{U}\mathbf{U}^t & \dfrac{1}{\dot{v}_e}\,\mathbf{U} \\ \dfrac{1}{\dot{v}_e}\,\mathbf{U}^t & \dfrac{v_o}{\dot{v}_e} \end{bmatrix}_{s=j\omega_i} \tag{34}$$

where a dot over v_e denotes the derivative of v_e with respect to s. To show that $\hat{\mathbf{K}}_i$ is nonnegative definite, we appeal to Lemma 1. Thus it is sufficient to point out that $(v_o/\dot{v}_e)\,|_{j\omega_i}$ is positive and that

$$\mathbf{K}_i + \frac{1}{\dot{v}_e v_o}\,\mathbf{U}\mathbf{U}^t \bigg|_{j\omega_i} - \frac{1}{\dot{v}_e^2}\,\mathbf{U}\mathbf{U}^t\,\frac{\dot{v}_e}{v_o}\bigg|_{j\omega_i} = \mathbf{K}_i \tag{35}$$

is nonnegative definite.

Finally, we must show that $\hat{\mathbf{K}}_\infty = [(1/s)\hat{\mathbf{Y}}]_\infty$ is nonnegative definite. When $k = 1$, the proof is trivial for then

$$\left[\frac{1}{sv_e}\,\mathbf{U}\right]_\infty = \left[\frac{1}{v_o}\,\mathbf{U}\right]_\infty = \mathbf{0}$$

and

$$\hat{\mathbf{K}}_\infty = \begin{bmatrix} \mathbf{K}_\infty & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} \tag{36}$$

† When $v = su = s(u_e + u_o)$, where u is a strict Hurwitz polynomial, it is necessary to replace v_e, v_o, and $\mathbf{U}$ respectively with u_o, u_e and the n-vector of even polynomials $s^{-1}\mathbf{U}$ before this argument is applied to verify the nonnegative definiteness of the matrix of residues associated with the pole at the origin.

When k exceeds unity,

$$\hat{\mathbf{K}}_\infty = \begin{bmatrix} \mathbf{K}_\infty + k\left[\frac{1}{v_o^2}\mathbf{U}\mathbf{U}^t\right]_\infty & k\left[\frac{1}{v_o}\mathbf{U}\right]_\infty \\ k\left[\frac{1}{v_o}\mathbf{U}^t\right]_\infty & k-1 \end{bmatrix}. \tag{37}$$

According to Lemma 1, $\hat{\mathbf{K}}_\infty$ is nonnegative definite if and only if

$$\begin{aligned} \mathbf{K}_\infty + k\left[\frac{1}{v_o^2}\mathbf{U}\mathbf{U}^t\right]_\infty - \frac{k^2}{k-1}\left[\frac{1}{v_o^2}\mathbf{U}\mathbf{U}^t\right]_\infty \\ = \mathbf{K}_\infty - \frac{k}{k-1}\left[\frac{1}{v_o^2}\mathbf{U}\mathbf{U}^t\right]_\infty \end{aligned} \tag{38}$$

is nonnegative definite. However, from Theorem 1 [condition(*iii*)] and the fact that

$$[\mathbf{Y}_e]_\infty = \left[\frac{1}{v_o^2}\mathbf{U}\mathbf{U}^t\right]_\infty \tag{39}$$

it follows that (38) is indeed nonnegative definite.

Therefore, the conditions of Theorem 1 are sufficient for the realization of $\mathbf{Y}(s)$. It is of interest to note that in the preceding constructive proof it was sufficient to assume that $\eta(s)$ [defined in Section IV] is unity. All other possible matrices $\hat{\mathbf{Y}}(s)$ corresponding to a realization of $\mathbf{Y}(s)$ can be generated by exploiting the permissible choices of $\eta(s)$.

To complete the theory we consider in the next section the cases in which $\hat{\mathbf{Y}}(s)$ does not exist or $\mathbf{Y}_e$ is a matrix of constants when $\hat{\mathbf{Y}}(s)$ does exist.

VI. NECESSARY AND SUFFICIENT CONDITIONS FOR THE REALIZATION OF $\mathbf{Y}(s)$ WHEN $\hat{\mathbf{Y}}(s)$ DOES NOT EXIST OR WHEN $\mathbf{Y}_e$ IS A MATRIX OF CONSTANTS

The results of this section for the case in which $\hat{\mathbf{Y}}(s)$ does not exist are based on the following result which is proved in Appendix B.

Lemma 2:

If $\mathbf{Y}(s)$ *in Fig. 1 exists but* $\hat{\mathbf{Y}}(s)$ *does not exist, then* v_{n+1}, *the voltage across the* RC *combination terminating port* $(n + 1)$, *is related to the other port voltages by*

$$v_{n+1} = \sum_{i=1}^{n} \beta_i v_i \tag{40}$$

where the β_i *are real constants.*

We wish to prove the following:

Theorem 2:

If the rational positive-real symmetric matrix $\mathbf{Y}(s)$, *defined by the structure in Fig. 1, exists but* $\hat{\mathbf{Y}}(s)$ *does not exist, or if* $\mathbf{Y}(s)$ *is such that* $\mathbf{Y}_e$ *is a matrix of constants,* $\mathbf{Y}(s)$ *can be expressed as* $s\mathbf{K}_\infty + \mathbf{K}_0 + \mathbf{Y}'(s)$ *where* $\mathbf{Y}'(s)$ *is an odd rational matrix in s such that* $\mathbf{Y}'(s) \rightarrow \mathbf{0}$ *as* $s \rightarrow \infty$, *and* $\mathbf{K}_0$ *is a real constant matrix with rank not exceeding unity such that* $\mathbf{K}_\infty - \mathbf{K}_0$ *is nonnegative definite. Further, if* $\mathbf{Y}(s)$ *satisfies the above condition, it can be realized as a reactance n-port in parallel with a network of ideal transformers that is terminated with a parallel combination of a unit resistor and a unit capacitor.*

To prove the theorem for the case in which $\hat{\mathbf{Y}}(s)$ does not exist, first consider the expression for P, the average power entering the n-ports defining $\mathbf{Y}(s)$, in terms of $\mathbf{Y}_e \mid_{s=j\omega}$ and $\mathbf{V}^t = [v_1, v_2, \cdots, v_n]$:

$$P = \mathbf{V}^t\mathbf{Y}_e \mid_{s=j\omega}\mathbf{V}^* \tag{41}$$

where the asterisk denotes the complex conjugate. Since P is also equal to $v_{n+1}v_{n+1}^*$, we have from (41) and Lemma 2

$$\mathbf{V}^t\mathbf{Y}_e \mid_{s=j\omega}\mathbf{V}^* = \sum_{i,j=1}^{n} \beta_i\beta_j v_i v_j^*. \tag{42}$$

Because (42) is valid for arbitrary v_i, we find

$$\mathbf{Y}_e \mid_{s=j\omega} = \mathbf{B}\mathbf{B}^t = \mathbf{K}_0 \tag{43}$$

where $\mathbf{B}^t = [\beta_1, \beta_2, \cdots, \beta_n]$. Thus $\mathbf{Y}(s)$ can be expressed as

$$\mathbf{Y}(s) = s\mathbf{K}_\infty + \mathbf{K}_0 + \mathbf{Y}'(s) \tag{44}$$

where $\mathbf{Y}'(s)$ is a matrix of odd rational functions which vanish at infinity. It is evident from (43) that $\mathbf{K}_0$ satisfies the rank conditions of Theorem 2.

Note that $\mathbf{Y}(s)$ has the form (44) if $\hat{\mathbf{Y}}(s)$ exists but $\mathbf{Y}_e$ is a matrix of constants; for, in this case also, the rank of $\mathbf{Y}_e$ cannot exceed unity.†

Next consider $\mathbf{Y}(s - 1)$, which must have a nonpositive definite real part on $s = j\omega$:

$$\mathbf{Y}(s - 1) = s\mathbf{K}_\infty + [\mathbf{K}_0 - \mathbf{K}_\infty] + \mathbf{Y}'(s - 1) \tag{45}$$

It is clear that if the real part of $\mathbf{Y}(s - 1)$ on $s = j\omega$ is to be nonpositive

† This is obvious from the form of (4).

definite for arbitrarily large values of $|\omega|$, $[\mathbf{K}_\infty - \mathbf{K}_0]$ must be a nonnegative definite matrix.†

Finally, assume that $\mathbf{Y}(s)$ satisfies the conditions of Theorem 2 and consider

$$\mathbf{Y}(s) = \mathbf{C}[s\mathbf{D}_\infty + \mathbf{D}_0]\mathbf{C}^t + \mathbf{Y}'(s) \tag{46}$$

in which the real nonsingular $n \times n$ matrix $\mathbf{C}$ is chosen so that

$$\begin{aligned} \mathbf{C}^{-1}\mathbf{K}_\infty\mathbf{C}^{-1t} &= \mathbf{D}_\infty \\ \mathbf{C}^{-1}\mathbf{K}_0\mathbf{C}^{-1t} &= \mathbf{D}_0 \end{aligned} \tag{47}$$

where $\mathbf{D}_\infty$ and $\mathbf{D}_0$ are diagonal matrices.‡ Note that there can be at most one nonzero term in $\mathbf{D}_0$ and that this term cannot exceed the corresponding entry in $\mathbf{D}_\infty$ for otherwise $[\mathbf{K}_\infty - \mathbf{K}_0]$ would not be nonnegative definite. Hence $\mathbf{Y}(s)$ can be rewritten as

$$\mathbf{Y}(s) = (s + 1)\mathbf{CFC}^t + \mathbf{Y}''(s)$$

where $\mathbf{Y}''(s)$ is realizable as a reactance network and $\mathbf{F}$ is a constant diagonal matrix with at most one nonzero element. This nonzero element is, of course, positive. The interpretation of the congruence transformation $\mathbf{CFC}^t$ in terms of an ideal transformer network is well known. This proves Theorem 2.

VII. SUMMARY AND RELATED REMARKS

The principal results can be summarized as follows.

Theorem 3:

The rational positive-real $n \times n$ symmetric short-circuit admittance matrix $\mathbf{Y}(s)$ is realizable as a lossless network containing inductors, capacitors, and ideal transformers and a two-terminal element comprising a parallel combination of a unit resistor and a unit capacitor if and only if

i. When $\mathbf{Y}_e = \mathbf{K}_0$, a matrix of constants, the rank of $\mathbf{K}_0$ does not exceed unity and $[(1/s)\mathbf{Y}]_\infty - \mathbf{K}_0$ is nonnegative definite.

ii. When $\mathbf{Y}_e$ is not a matrix of constants, (a) $\mathbf{Y}_e$ can be expressed in

† This conclusion, with $\mathbf{K}_0$ defined as $[\mathbf{Y}_e]_\infty$, is valid when any number of unit resistor-capacitor parallel combinations are imbedded in a general lossless network.

‡ This is always possible since $\mathbf{K}_\infty$ and $\mathbf{K}_0$ are nonnegative definite.

standard form (defined in Section III) with $v(s)$ *such that* $k = [v_o/sv_e]_\infty \geqq 1$; (*b*) *if* $k = 1$, $[\mathbf{Y}_e]_\infty = \mathbf{0}$; (*c*) *if* $k > 1$, $\left[\frac{1}{s}\mathbf{Y}\right]_\infty - \frac{k}{k-1}[\mathbf{Y}_e]_\infty$ *is nonnegative definite.*

Further, if $\mathbf{Y}_e$ *is a matrix of constants and satisfies condition* (*i*), $\mathbf{Y}$ *can be realized as a reactance n-port in parallel with a network of ideal transformers that is terminated with a parallel combination of a unit resistor and a unit capacitor. If* $\mathbf{Y}_e$ *is not a matrix of constants and satisfies condition* (*ii*), $\mathbf{Y}$ *can be realized as an* $(n+1)$*-port lossless network, characterized by the short-circuit admittance matrix* $\hat{\mathbf{Y}}$, *terminated at port* $(n+1)$ *with a parallel combination of a unit resistor and a unit capacitor. The matrix* $\hat{\mathbf{Y}}$ *is given by*

$$\hat{\mathbf{Y}} = \begin{bmatrix} \mathbf{Y} + \dfrac{1}{v_e(v_o + v_e)}\,\mathbf{U}\mathbf{U}^t & \dfrac{1}{v_e}\mathbf{U} \\[2ex] \dfrac{1}{v_e}\mathbf{U}^t & \dfrac{v_o}{v_e} - s \end{bmatrix}.$$

For completeness, we state the following extension of Theorem 3.

Theorem 4:

The short-circuit admittance matrix $\bar{\mathbf{Y}}(s)$ *is realizable as a lossless network containing inductors, capacitors, and ideal transformers and a two-terminal element comprising a parallel combination of a resistor of value R ohms* $(R > 0)$ *and a capacitor of value* TR^{-1} *farads* $(T > 0)$ *if and only if* $\mathbf{Y}(s) = \bar{\mathbf{Y}}(s/T)$ *is a symmetric positive-real matrix that satisfies the conditions of Theorem 3. If instead the resistor is equal to* $-R$ *ohms, the matrix is realizable if and only if* $\mathbf{Y}(s) = -\bar{\mathbf{Y}}(-s/T)$ *is a symmetric positive-real matrix that satisfies the conditions of Theorem 3.*

The proof of Theorem 4 follows from two elementary transformations and is omitted.† In each case the parameter T is, of course, the time constant of the RC combination. It is convenient for some purposes to have the realizability conditions stated explicitly in terms of T. This can easily be done with the aid of the above theorem and is discussed in Appendix C.

† The fact that an $n \times n$ short-circuit admittance matrix $\mathbf{Y}(s)$ of real rational functions is realizable as a network containing only lossless elements and negative resistors if and only if $-\mathbf{Y}(-s)$ is a positive-real matrix was first established by Carlin and Youla.[9]

The following theorem states an interesting inequality involving $[(1/s)\mathbf{Y}]_\infty$ and $[(1/s)\mathbf{Y}_{sc}]_\infty$ where $\mathbf{Y}_{sc}$, if it exists, is the value of $\mathbf{Y}(s)$ when the RC combination, with unit capacitance, is shorted.

Theorem 5:

$\left[\frac{1}{s}\mathbf{Y}\right]_\infty - \frac{1}{k}\left[\frac{1}{s}\mathbf{Y}_{sc}\right]_\infty$ *is nonnegative definite.*

The proof follows at once from (21) and the fact that (24) is nonnegative definite.

The following theorem is of assistance in simplifying the tests indicated in Theorem 3 for the important case in which $\mathbf{K}_\infty = [(1/s)\mathbf{Y}]_\infty$ is positive definite.

Theorem 6:

If $\mathbf{A}$ *and* $\mathbf{B}$ *are* $n \times n$ *real symmetric nonnegative definite matrices with det* $\mathbf{A} \neq 0$ *and* $\mathbf{B}$ *of unit rank,* $\mathbf{A} - \mathbf{B}$ *is nonnegative definite if and only if det* $[\mathbf{A} - \mathbf{B}] \geqq 0$.

To prove this result note that $\mathbf{A} - \mathbf{B}$ can be written as $\mathbf{Q}[\mathbf{D}_a - \mathbf{D}_b]\mathbf{Q}^t$, where $\mathbf{Q}$ is a real nonsingular matrix such that $\mathbf{A} = \mathbf{Q}\mathbf{D}_a\mathbf{Q}^t$, $\mathbf{B} = \mathbf{Q}\mathbf{D}_b\mathbf{Q}^t$, and $\mathbf{D}_a$ and $\mathbf{D}_b$ are diagonal matrices. Thus

$$\det[\mathbf{A} - \mathbf{B}] = \det{}^2\mathbf{Q}\cdot\det[\mathbf{D}_a - \mathbf{D}_b].$$

The realizability conditions can be expressed also in terms of the open-circuit impedance matrix $\mathbf{Z}(s)$ by exploiting an approach similar to that used in treating the short-circuit admittance matrix $\mathbf{Y}(s)$. Since the ideas involved are so similar to those discussed earlier we shall omit the details and simply state the result:

Theorem 7:

The rational positive-real $n \times n$ *symmetric open-circuit impedance matrix* $\mathbf{Z}(s)$ *is realizable as a lossless network containing inductors, capacitors, and ideal transformers and a two-terminal element comprising a parallel combination of a unit resistor and a unit capacitor if and only if the even part of the matrix* $\mathbf{Z}$, $\mathbf{Z}_e$, *is expressible in standard form (defined in Section III) with*

$$\left[\frac{1}{s}\frac{v_e}{v_o}\right]_\infty \geqq 1\ .$$

Further, if $v_e \not\equiv sv_o$ *and* $\mathbf{Z}$ *satisfies the above conditions,* $\mathbf{Z}$ *can be realized*

as an $(n+1)$-port lossless network, characterized by the open-circuit impedance matrix $\hat{\mathbf{Z}}$, *terminated at port $(n+1)$ with a parallel combination of a unit resistor and a unit capacitor. The matrix* $\hat{\mathbf{Z}}$ *is given by*

$$\hat{\mathbf{Z}} = \begin{bmatrix} \mathbf{Z} + \mathbf{U}\mathbf{U}^t \dfrac{(s+1)}{(v_e - sv_o)(v_o + v_e)} & \mathbf{U}\dfrac{1}{v_e - sv_o} \\ \mathbf{U}^t \dfrac{1}{v_e - sv_o} & \dfrac{v_o}{v_e - sv_o} \end{bmatrix}.$$

If $v_e \equiv sv_o$, *and* $\mathbf{Z}$ *satisfies the above conditions,* $\mathbf{Z}$ *can be expressed as* $\mathbf{Z} = [1/(s+1)]\,\mathbf{F} + \mathbf{Z}'$ *where* $\mathbf{F}$ *is a real symmetric nonnegative definite matrix of constants of rank not exceeding unity and* $\mathbf{Z}'$ *is the open-circuit impedance matrix of an n-port reciprocal lossless network.*

The simple form that the conditions assume is attributable to the fact that the impedance of the parallel RC combination is regular at infinity and that the matrix $\mathbf{Z}_e$ is not a matrix of constants unless every element vanishes identically in s.

APPENDIX A

Factorization of $\mathbf{Y}_e(s)$

Recall that $\mathbf{Y}_e$ is the even part of a rational symmetric $n \times n$ positive-real short-circuit admittance matrix. It is convenient to partition this matrix as follows:

$$\mathbf{Y}_e(s) = \begin{bmatrix} E_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{12}{}^t & \mathbf{E}_{22} \end{bmatrix} \tag{48}$$

where E_{11}, $\mathbf{E}_{12}$, and $\mathbf{E}_{22}$ are respectively 1×1, $1 \times (n-1)$, and $(n-1) \times (n-1)$ submatrices of ratios of even polynomials in s. We may assume, without loss of generality, that E_{11} does not vanish identically in s.

Consider the following identity which is readily verified:

$$\begin{bmatrix} E_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{12}{}^t & \mathbf{E}_{22} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{E}_{12}{}'E_{11}{}^{-1} & \mathbf{1}_{n-1} \end{bmatrix} \begin{bmatrix} E_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_{22} - \mathbf{E}_{12}{}'\mathbf{E}_{12}E_{11}{}^{-1} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{E}_{12}E_{11}{}^{-1} \\ \mathbf{0} & \mathbf{1}_{n-1} \end{bmatrix}. \tag{49}$$

It is evident that the normal rank of $\mathbf{Y}_e$ cannot exceed unity if it is to be expressible in standard form. Accordingly we may assume that

$\mathbf{E}_{22} - \mathbf{E}_{12}{}^t\mathbf{E}_{12}E_{11}{}^{-1} = \mathbf{0}$, and hence

$$\begin{bmatrix} E_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{12}{}^t & \mathbf{E}_{22} \end{bmatrix} = E_{11}[1 \quad \mathbf{E}_{12}E_{11}{}^{-1}]^t[1 \quad \mathbf{E}_{12}E_{11}{}^{-1}]. \tag{50}$$

The right-hand side of (50) can be rewritten as $f\mathbf{P}\mathbf{P}^t$ where

$$\mathbf{P}^t = [p_1, p_2, \cdots, p_n]$$

is a row matrix of even real polynomials and f is an even real rational fraction in s, analytic on $s = j\omega$ $[-\infty \leqq \omega \leqq \infty]$, and such that

$$f(j\omega) \geqq 0.$$

As is well known, $f(s)$ can be expressed as either

$$\frac{g^2}{h(s)h(-s)} \qquad \text{or} \qquad \frac{-l^2}{m(s)m(-s)}$$

where g and l are respectively even and odd real polynomials and $h(s)$ and $m(s)$ are real strict Hurwitz polynomials. In either case, since

$$\frac{g^2}{h(s)h(-s)} = \frac{-(sg)^2}{[sh(s)][-sh(-s)]},$$

$\mathbf{Y}_e$ can be written as

$$\mathbf{Y}_e = \frac{-1}{w(s)w(-s)} \mathbf{W}\mathbf{W}^t,$$

in which $\mathbf{W}^t = [w_1, w_2, \cdots, w_n]$ is a row matrix of real odd polynomials and $w(s)$ is a real strict Hurwitz polynomial except possibly for a simple zero at the origin.

Note that $\mathbf{Y}$ is realizable as shown in Fig. 1 when $\hat{\mathbf{Y}}$ exists only if the degree of $w(s)$ is odd.

APPENDIX B

Proof of Lemma 2

First note that if the $(n + 1)$-port lossless network does not possess a short-circuit admittance matrix, the short-circuit admittance matrix of $\bar{N}$, the lossless $(n + 1)$-port with a unit capacitor added in parallel at port $(n + 1)$, also does not exist.

Let $\bar{\mathbf{S}}(s)$ be the scattering matrix[10] of $\bar{N}$ and consider the circuit

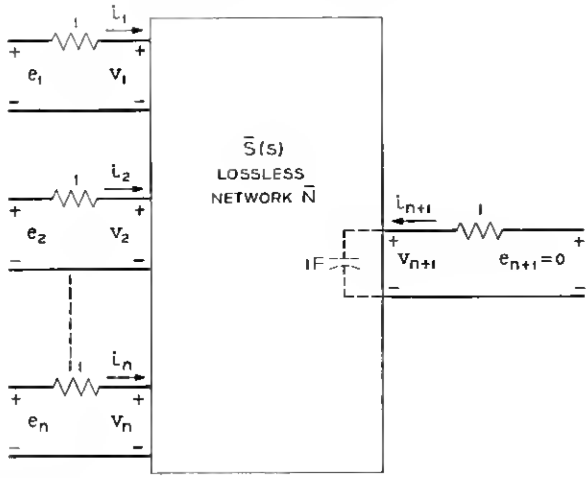

Fig. 2 — Network defining the relationship between **E**, **V**, **I**, and the scattering matrix $\bar{\mathbf{S}}(s)$.

shown in Fig. 2. By definition

$$\bar{\mathbf{S}}[\mathbf{V} + \mathbf{I}] = \mathbf{V} - \mathbf{I}, \tag{51}$$

and

$$\mathbf{Y}(s) = (\mathbf{1}_n + \mathbf{S})^{-1}(\mathbf{1}_n - \mathbf{S}) \tag{52}$$

where **S** is the matrix of elements in the first n rows and columns of $\bar{\mathbf{S}}$.

Substituting $\mathbf{E} = \mathbf{V} + \mathbf{I}$ in (51) gives

$$\mathbf{V} = \tfrac{1}{2}[\bar{\mathbf{S}} + \mathbf{1}_{n+1}]\mathbf{E}. \tag{53}$$

Because the short-circuit admittance matrix of $\bar{N}$ exists if and only if $\det[\bar{\mathbf{S}} + \mathbf{1}_{n+1}] \not\equiv 0$, and since **S** is the matrix of elements in the first n rows and columns of $\bar{\mathbf{S}}$, it follows that $[\bar{\mathbf{S}} + \mathbf{1}_{n+1}]$ has normal rank† equal to n. Further, since the rank of $[\bar{\mathbf{S}} + \mathbf{1}_{n+1}]$ is invariant in the strict right-half plane, there exists, to within an arbitrary scalar multiplicative factor, one and only one real constant $(n + 1)$-vector **X** such that

$$[\bar{\mathbf{S}}(s_0) + \mathbf{1}_{n+1}]\mathbf{X} = \mathbf{0} \tag{54}$$

where s_0 is a fixed but arbitrarily chosen real positive constant. Let **X** be normalized so that $\mathbf{X}'\mathbf{X} = 1$. Equation (54) then yields

$$\mathbf{X}'\bar{\mathbf{S}}(s_0)\mathbf{X} = -1 \tag{55}$$

† Since $\mathbf{Y}(s)$ exists, $\det[\mathbf{1}_n + \mathbf{S}]$ does not vanish identically in s.

Note that $\mathbf{X}^t\overline{\mathbf{S}}(s)\mathbf{X}$ is a one-port passive scattering coefficient and that therefore (51) implies

$$\mathbf{X}^t[\overline{\mathbf{S}}(s) + \mathbf{1}_{n+1}]\mathbf{X} = 0 \tag{56}$$

identically in s. Furthermore, since $[\overline{\mathbf{S}}(s) + \mathbf{1}_{n+1}]$ is positive semidefinite for all real positive s, it follows that

$$\mathbf{X}^t[\overline{\mathbf{S}}(s) + \mathbf{1}_{n+1}] = \mathbf{0} \tag{57}$$

identically in s. Thus from (53) and (57)

$$\mathbf{X}^t\mathbf{V} = \tfrac{1}{2}\mathbf{X}^t[\overline{\mathbf{S}} + \mathbf{1}_{n+1}]\mathbf{E} = 0$$

or

$$\sum_{i=1}^{n+1} x_i v_i = 0$$

where the x_i are real constants, not all zero. However since $\mathbf{Y}(s)$ exists x_{n+1} cannot vanish. Dividing through by x_{n+1} gives an expression of the form

$$v_{n+1} = \sum_{i=1}^{n} \beta_i v_i$$

in which the β_i are real constants.

It is of incidental interest to note that the proof does not require that $\overline{N}$ be lossless. It is sufficient that it be passive.†

APPENDIX C

The Realizability Conditions Stated Explicitly in Terms of the Parameter T

According to Theorem 4, $\overline{\mathbf{Y}}(s)$ is realizable with a lossless reciprocal network and a two-terminal element comprising a parallel combination of a resistor of value R ohms ($R > 0$) and a capacitor of value TR^{-1} farads ($T > 0$) if and only if $\mathbf{Y}(s) = \overline{\mathbf{Y}}(s/T)$ is a symmetric positive-real matrix that satisfies the conditions of Theorem 3. Let $\overline{\mathbf{Y}}_e(s)$ be expressed in standard form:

$$\overline{\mathbf{Y}}_e(s) = \frac{-1}{\bar{v}(s)\bar{v}(-s)}\overline{\mathbf{U}}\overline{\mathbf{U}}^t. \tag{58}$$

† The original version of this proof, based also on the formulation (53) and the fact that $[\overline{\mathbf{S}} + \mathbf{1}_{n+1}]$ is of normal rank n, assumed that the network $\overline{N}$ is lossless and hence that $\overline{\mathbf{S}}(s)$ is a regular para-unitary matrix. The final version of the proof was suggested by D. C. Youla.

In terms of $\bar{v}(s)$,

$$k = \left[\frac{1}{s}\frac{\bar{v}_o(s/T)}{\bar{v}_e(s/T)}\right]_\infty = \frac{1}{T}\bar{k} \tag{59}$$

$$\bar{k} = \left[\frac{1}{s}\frac{\bar{v}_o(s)}{\bar{v}_e(s)}\right]_\infty .$$

Also,

$$\left[\frac{1}{s}\mathbf{Y}(s)\right]_\infty = \left[\frac{1}{s}\overline{\mathbf{Y}}\left(\frac{s}{T}\right)\right]_\infty = \frac{1}{T}\left[\frac{1}{s}\overline{\mathbf{Y}}(s)\right]_\infty ,$$

and

$$[\mathbf{Y}_e(s)]_\infty = \left[\overline{\mathbf{Y}}_e\left(\frac{s}{T}\right)\right]_\infty = [\overline{\mathbf{Y}}(s)]_\infty .$$

Theorem 3 and the above equations yield

Theorem 8:

The rational positive-real $n \times n$ symmetric short-circuit admittance matrix $\overline{\mathbf{Y}}(s)$ is realizable as a lossless network containing inductors, capacitors, and ideal transformers and a two-terminal element comprising a parallel combination of a resistor of value R ohms ($R > 0$) and a capacitor of value TR^{-1} Farads ($T > 0$) if and only if

i. When $\overline{\mathbf{Y}}_e(s) = \overline{\mathbf{K}}_0$, a matrix of constants, the rank of $\overline{\mathbf{K}}_0$ does not exceed unity and $[(1/s)\overline{\mathbf{Y}}(s)]_\infty - T\overline{\mathbf{K}}_0$ is nonnegative definite.

ii. When $\overline{\mathbf{Y}}_e(s)$ is not a matrix of constants (a) $\overline{\mathbf{Y}}_e(s)$ can be expressed in standard form with $\bar{k} \geqq T$; (b) if $\bar{k} = T$, $[\overline{\mathbf{Y}}_e(s)]_\infty = \mathbf{0}$; (c) if $\bar{k} > T$,

$$\left[\frac{1}{s}\overline{\mathbf{Y}}(s)\right]_\infty - \frac{\bar{k}T}{\bar{k} - T}[\overline{\mathbf{Y}}_e(s)]_\infty$$

is nonnegative definite.

Similarly, Theorem 5 can be transformed to read:

$$\left[\frac{1}{s}\overline{\mathbf{Y}}(s)\right]_\infty - \frac{T}{\bar{k}}\left[\frac{1}{s}\overline{\mathbf{Y}}_{sc}(s)\right]_\infty$$

is nonnegative definite.

The modifications necessary to treat the case in which the resistance is equal to $-R$ ohms are obvious in view of Theorem 4.

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